REAL NUMBERS AND THEIR INTERESTING PROOFS

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ABSTRACT : This research aims all about the real numbers, the other number systems included in it and some of the important proofs, along with some interesting and important applications in real life as well as in mathematics. Real numbers are the numbers which are represented by the number line. This number line is similar to an ideal ruler, with which it is possible to measure the lengths of segments in Euclidean geometry. Due to invention of real numbers, invention of calculus came into existence, so real numbers are important in understanding the entire universe. The properties of real numbers fall into 3 categories: algebraic properties, order properties and completeness property. Completeness property defines that real number line is complete, i.e., there are no gaps or holes in it. The set of real numbers and other number systems have same properties, but the completeness property helps us to distinguish the set of $\mathbb{R}$ from other sets. Using Dedekind cuts', the properties of real numbers are established. A detailed account of construction of real numbers was beginning with Peano postulates and using the Dedekind methods to introduce irrational numbers. The most difficult process of construction of real numbers is transition of rational numbers toreal numbers. The representation of real numbers in decimal expansion and binary expansion is also covered in this paper.

Keywords: Real numbers, p-adic numbers, transcendental numbers, applications

INTRODUCTION

The concept of Real numbers is vast in Mathematics. There are many interesting proofs and applications of Real numbers. Real numbers are denoted by$\mathbb{R}$. It consists of Natural numbers, Integers, Rational numbers and Irrational numbers. Real numbers satisfy Closure property, Commutative property, Associative property, Identity property and Inverse property. Real numbers can be added, subtracted, multiplied and divided (except 0) to produce more real numbers under the rules of arithmetic with the help of these algebraic properties. The algebraic properties are often called field properties in abstract algebra that are based on two binary operations of addition and multiplication, in order properties of$\mathbb{R}$, we derive consequences of these properties and illustrates their use in working with inequalities and in final the completeness property which is used in theory of limits and continuity and to derive several fundamental results concerning$\mathbb{R}$. Hence, Real numbers is a setconsists of rational and irrational numbers.

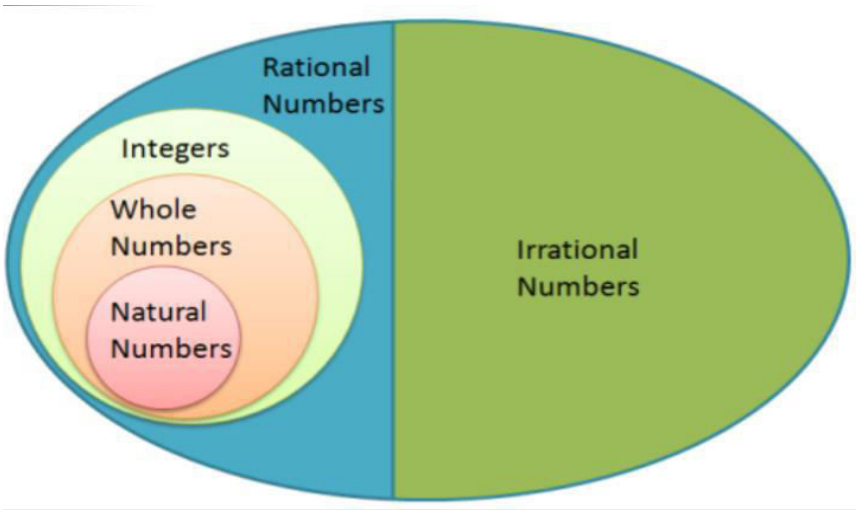


Figure 1: Real Number System

Geometrically, Real numbers are represented on a number line. Integers are represented as points on line with fixed point '0' on it and fixed unit distance. 0 is at center on the number line, at right of 0 there are positive integers and negative integers are on left side. There also exist rational numbers on number line, they spread themselves densely on line. But still there remains a gap, in which there exists irrational numbers. Real numbers are never ending. So, on corner of number line, the symbol as plus infinity(+∞) and minus infinity (-∞) is used, which denotes that these numbers on number line never ends. Infinity (∞) is not a number; it is a symbol.

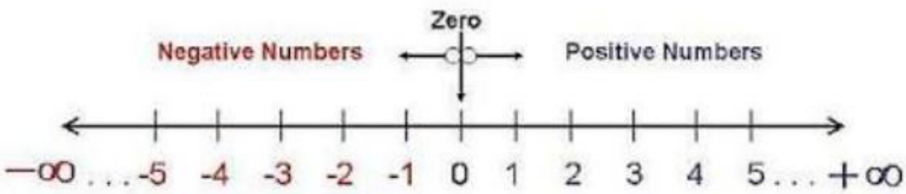


Figure 2: Real Line (Sudhir&Balmohan, 2006)

All real numbers are represented in form of decimal expansion, which consists of finite and infinite number of digits and decimal points. Decimal representation of real numbers is similar to binary representation. Let x be any real number then decimal expansion of x is, x = b.$a_1a_2a_3$....= $b + \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots$,where a_i are integers and $0 \le a_i \le 9$. This expansion is unique except for x =$\frac{1}{2}$, which can be expanded as, x = 0.50000… and x = 0.499999....

This is decimal representation of real numbers, whereas binary representation of real numbers consists of only digits 0 and 1. For example, x = 0.$a_1a_2a_3$……. means $\frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots$such that $\frac{1}{2}$=0.100…, $\frac{1}{4}$= 0.0100…, $\frac{1}{16}$ = 0.0001…. we can write = $\frac{13}{16}$ in form of binary expansion $\frac{13}{16} = \frac{1}{2} + \frac{1}{4} + \frac{1}{16}$= 0.110100….

Similarly, the ternary expansion of real number x consists only digits 0, 1 and 2. For example, x = 0.$a_1a_2a_3$…… means x = $\frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3}$……… such that $\frac{1}{3}$ = 0.10000….and $\frac{1}{2}$ = 0.1111..., hence,$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$ = 0.2111….

The ternary expansion for a real number x is unique except for numbers such as $\frac{1}{3}$ with two expansions, one ending in a string of 2’s and the other in a string of 0’s. (Goldberg, 1970)

There are many proofs and interesting inventions on real numbers. Some of the theorems of real numbers are also used to prove some properties of other concepts in mathematics. There are also some of special constants in real numbers such as pi(π), Euler’s number(e),$\sqrt{2}$, $\sqrt{3}$, we will also see their existence on number line.

HISTORY OF REAL NUMBERS

The system of Real numbers came into existence by successive extension of the system of natural numbers. Natural numbers came into existence when man first learnt counting. Addition of two natural numbers get a natural number, but inverse operation of subtraction is not possible. In order that operation of subtraction can to be performed without any restriction, it became necessary to extend the system of natural numbers, introduction of negative numbers and zero came into existence. Thus, for every natural number ‘n’ there exists a unique negative number ‘-n’. Therefore, a new number system came into existence,i.e., Integers which consists of negative numbers, positive numbers and zero. Similarly, to make division possible, concept of fractions was introduced. So, the new system was introduced that is rational numbers which consist Integers and fractions. But there were lengths which could not be measured in the terms of rational numbers, like length of diagonal of square whose sides are of unit length were unable to measure in terms of rational numbers. Therefore, rational numbers have to be enlarged and Irrational numbers came into existence such as √2, π, e, etc. In this way real number system was formed. (Thomas& Finney, 1998)

Many mathematicians used real numbers to discover new proofs and inventions, but the invention of real numbers was first discovered by German Mathematician Julius Wilhelm Richard Dedekind. He introduced 'Dedekind cuts' as an approach towards real numbers in 'Continuity and Irrational numbers'. Dedekind cut is a partition set of rational numbers in two nonempty subsets say, A and B, such that all elements of A are less than elements in B. The corresponding cuts among rational numbers defines real numbers uniquely. Dedekind cuts are used to work on incomplete number sets. Due to these cuts, many real numbers came into existence. (Malik & Arora, 1984)

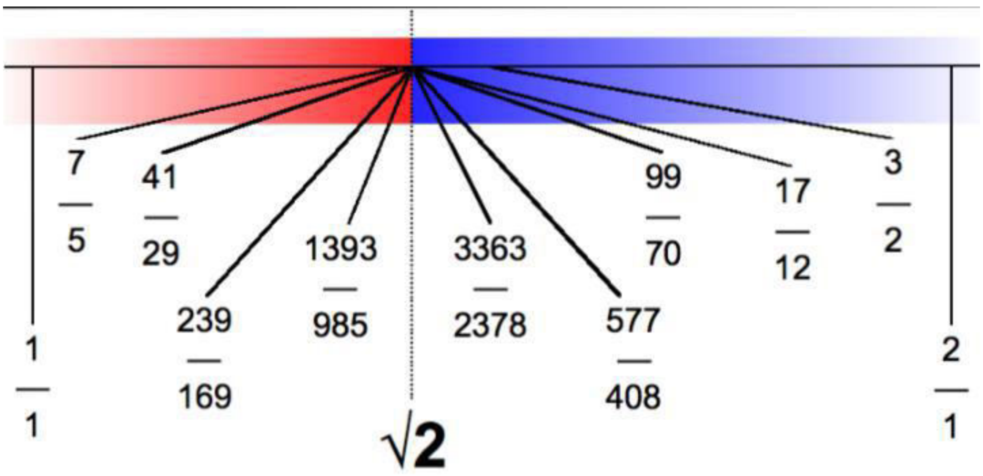


Figure 3: Dedekind cut (Malik and Arora, 1984)

The above figure is of Dedekind cut; in this figure we have found the existence of $\sqrt{2}$ on number line. Similarly, we can find many irrational numbers by using Dedekind cut method.

OTHER NUMBER SYSTEMS

Real numbers consist of natural numbers, integers, rational and irrational numbers. There are many useful properties and applications of these numbers.

i) NATURAL NUMBERS:

Natural numbers are positive numbers greater than 0. This numbers are used for counting. It is denoted by $\mathbb{N}$. Set of natural numbers is given by, $\mathbb{N}$={1,2,3,4…}. The set of natural number is infinite.

Natural numbers play an important role in sequence and series, it is a domain set in sequence and series. Here, the definition of convergent sequence says, A sequence (x_n) is said to converge to the limit ℓ if and only if the following criterion is satisfied, if given any $\varepsilon > 0$, we can find an N such that, for any n >N, | $x_n - \ell$ | < ε, we also write $x_n \rightarrow \ell$ as $n \rightarrow \infty$ or $\lim_{n\to\infty} x = \ell$. Convergence of a sequence is explained by a diagram, let $x_1, x_2, x_3 \ldots$ be the successive approximation to the number ℓ.The distance $|x_n - \ell|$ between x_n and ℓ is the error involved in approximating ℓ by x_n. The definition of convergence simply asserts that the error has to be small as by taking n large enough. For the value of ε indicated in the diagram, a suitable value for N in the definition is N = 6 for each value of n >6, $| x_n - \ell | < \varepsilon$

In particular, the value of $| x_9 - \ell |$ has been noted in the diagram.

The definition of convergence begins 'Given any ε > 0, to find an N, the emphasis is on the very small values of ε>0. It is clear from the diagram that, for very much smaller value of ε >0, there is need to pick out a very much larger value of N.

In general, smaller the value of ε>0, the bigger must be the corresponding value of N.

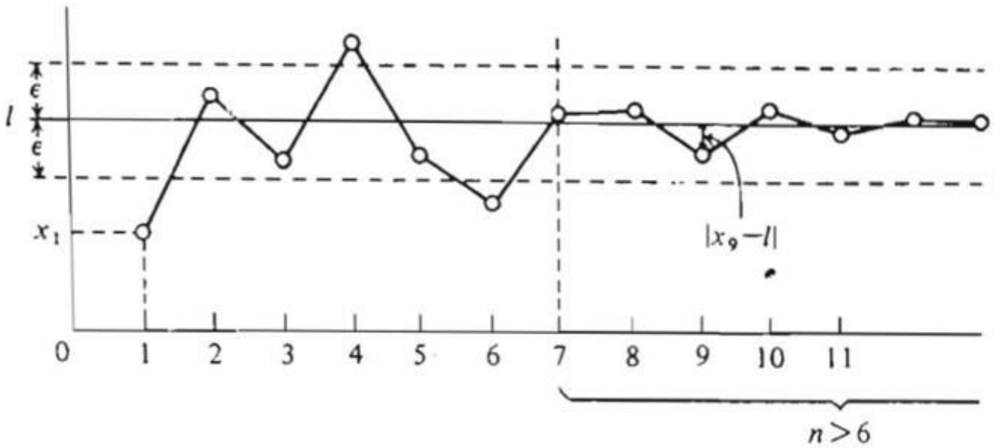


Figure 4: Representation of a sequence (x_n) (Binmore, 1977)

The above diagram represents a sequence (x_n) with the property that, as $x_n \rightarrow l$ implies $n \rightarrow \infty$. (Binmore, 1977)

ii) INTEGERS:

Integers consist of positive numbers, negative numbers and 0. It is denoted by $\mathbb{Z}$. Set of Integers is given by, $\mathbb{Z}$={…. -3, -2, -1,0,1,2,3….}.Integers are represented on number line.

Integer Zero (0)

One of the most interesting and important invention inMathematicsis invention of Zero.Mathematician Brahmagupta in AD 628 first time defined Zero. Zero is an important integer in the number system. Zero is an integer as well as a rational and real number. Zero is neither positive nor negative, it is placed at center of number line. It is additive identity of real numbers. Zero is an even number as it is divisible by 2 with no remainder. The expression (0/0) is very difficult to obtain, but the value of such expression can be obtained by LHospital's rule which is obtained by limit of [f(x)/g(x)], and if this limit exists, the value of this expression is obtained and if limit does not exist, the expression has to be solved by another method. (Sudhir&Balmohan, 2006)

iii) RATIONAL NUMBERS:

Rational numbers consist of Integers and fractions. It is denoted by '$\mathbb{Q}$'. Set of rational numbers is given by, $\mathbb{Q} = \{\frac{m}{n}$:m,n$\in\mathbb{Z}$; n$\neq$0$\}$. The set of rational numbers has all algebraic and order properties of real numbers but lacks the completeness property i.e there is a hole in rational line and this hole is filled by irrational numbers but there is a different number system which is the expansion of ordinary arithmetic of rational numbers. It is known as p-adic number system.

P- adic numbers

Kurt Hensel (1861-1941), the German mathematician, discovered the p-adic numbers around the end of 19th century. This number system is used intensively in number theory. This numbers were discovered around 100 years ago, but still there is an aura of mystery. Here the letter 'p' stands for fixed prime. The p-adic numbers were discovered by an attempt to bring ideas and techniques about power series methods in number theory. The compact field $\mathbb{Q}_p$ of p- adic numbers is a completion of rational numbers. The p-adic

$$r = \sum_{i=k}^{\infty} a_i p^i$$

expansion of rational numbers are in the form of power series, where $0 \le a_i \le p-1$

Geometrically p-adic numbers are represented in models named Euclidean model, one of the Euclidean model of p adic number is Model of Z_3: SierpinskyGasket. In this model, 'V' is the Euclidean space, namely finite dimensional inner product space over the field of $\mathbb{R}^3$ of real numbers, here p=3 with canonical basis e_o, e_1, e_2 and v(k) = e_k, then the corresponding vector maps $\psi:\mathbb{Z}_3 \to \mathbb{R}^3$. In this case, the image of ψ is contained in the plane x+y+z = 1. Since the components of the images ψ(a) are positive, the image of the map ψ contained in the unit simplex of $\mathbb{R}^3$ (convex span of the basic vectors).More precisely, the mappings ψ are injective for b > 2, and hence give homomorphic models of $\mathbb{Z}_3$ in this simplex.

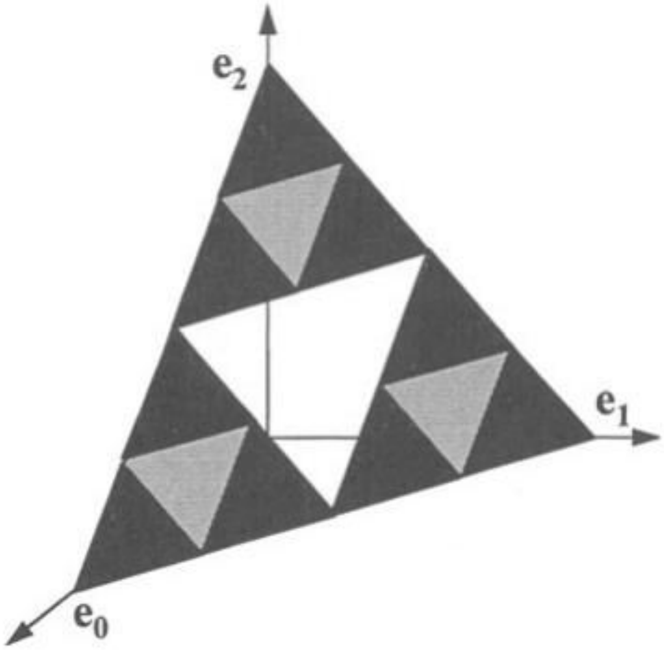


Figure5 :Model of Z_3 : Sierpitiski Gasket (Alain, 2000)

When b = 2, the image is a Sierpitiski gasket, hence connected in this simplex. In general, the image is a fractal having self-similarity dimension log 3/ log b**.** (Alain, 2000)

iv) IRRATIONAL NUMBERS:

All the real numbers except rational numbers are Irrational numbers. It is denoted by ($\mathbb{R}\backslash\mathbb{Q}$). These numbers are decimal expansion that never ends nor repeats.

<u>Existence of irrational numbers</u>

Pi (π)

Pi(π) is a mysterious number in real numbers. It is an Irrational number. It cannot be expressed in form of common fraction, but its approximate value is$\frac{22}{7}$or 3.14592…. and so on. Mathematician William Jones in 1706 used Greek letter π to represent the ratio of a circle's circumference to its diameter. Pi (π) is also known as Archimedes constant, because he created an algorithm to approximate the value of π. He was the first mathematician to calculate the theoretical value of π. He calculated it, by inscribing and circumscribing polygons on a circle. He used Euclid's theorem for faster calculation.

Manymathematicians started to calculate the value of π up to certain digits with the help of geometric techniques, infinite series etc. After the calculation of hundreds of digits of π, invention of calculus came into existence.

Figure6: Pi(π)(Petr, 1970)

Pi appears in many formulas of mathematics and physics. π is also known as transcendental number, it means it does not have any solution to polynomial equation whose coefficients are rational.

Many mathematicians tried to find the value of pi(π) by using concepts of Euclidean geometry, Infinite series, etc. In Euclidean geometry, π is defined as ratioof circumference of circle to its diameter.

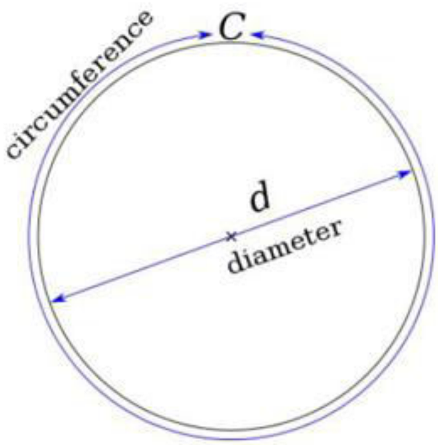


Figure7: Circle(Petr, 1970)

π is commonly defined as ratio of circumference of circle (C) to its diameter (d)

$\pi = \frac{c}{d}$

There are many applications of π in many mathematical concepts like in Euclidean geometry, trigonometry, building and constructions, science, astronomy, etc. (Petr, 1970)

Euler's number (e)

The number e is known as Euler's number and is approximately equal to 2.718338…. It is a base of natural logarithm. The two ways of calculating the value of e is, calculating the limit of $(1 + \frac{1}{n})^n$ when n → ∞and also by calculating the series, $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ….$

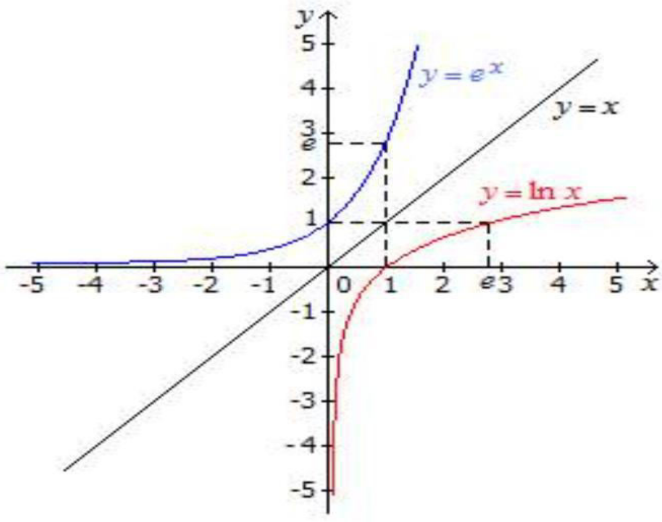


Figure8: Graph of exponential function (Sudhir&Balmohan, 2006)

The exponential function $f(x) = e^x$, it is a unique function which is its own derivative and also itis its own antiderivative. If x = 1, we getf (1) = e^x.

Euler's number is used in statistics, in probability theory, in many random distributions, also used in problems based on derangement, in derivations of physics, etc. Like the constant π, e is a transcendental number.(Sudhir&Balmohan, 2006)

APPLICATIONS

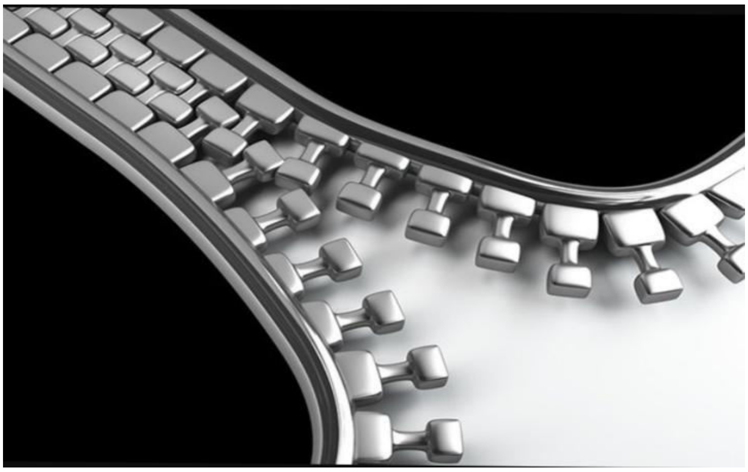
There are many applications of Real numbers in mathematics as well as in our day-to-day life

1) Real numbers are used to measure quantities of different objects

2) Natural numbers are used for counting, for ordering and for defining other concepts in mathematics. The uses of natural numbers are endless in life.
3) The principle of mathematical induction of natural numbers is used to prove that if dominoes are arranged in a manner given below, if first one falls then all dominoes will fall.

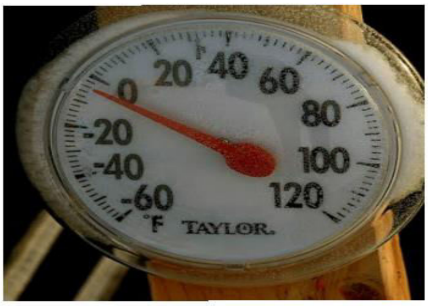
(David 2011)

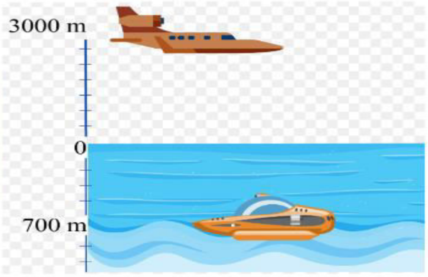
4) This principle is also used to prove that we can successfully zip a proper zipper if the first teeth of zip is zipped successfully.

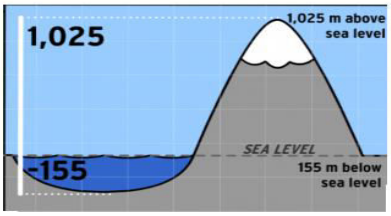


(David 2011)

5) Integers are commonly used in describing temperature above/below freezing point, blood pressure, a geographical level above/below sea level, elevator level when it is above/below the ground level

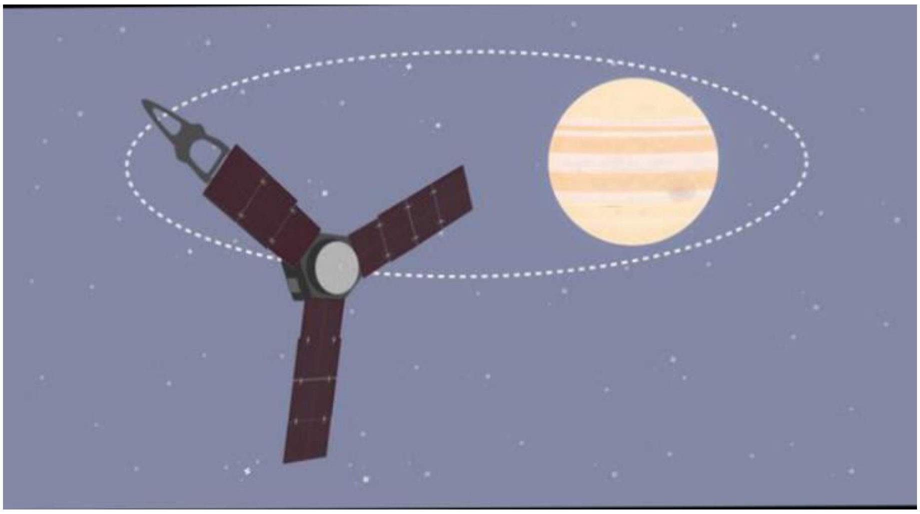






(Jim 2017)

6) Astronomers useπ to measure a planets size. Scientists useπ to put spacecraft into orbit around other planets. Using π, they can compute exactly how much they need to put on the brakes means firing forward-facing thrusters of spacecraft at just the right moment. (John 1992)



CONCLUSION

In this project, there is all about real numbers, their properties and their interesting applications in day-to-day life as well as in mathematics. Also studied about some special constants, representation of real numbers on the number line and studied about a very interesting number system i.e., p-adic numbers. The number system included in real numbers are important because they can give more information about the problems in real life. Real numbers are important in our life.Real numbers have given contribution in many mathematical concepts, due to real numbers we have many such interesting and important proofs and theorems in mathematics.

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